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CONCERNING ARCWISE CONNECTEDNESS AND THE EXISTENCE OF SIMPLE CLOSED CURVES IN PLANE CONTINUA

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1. **Introduction.** If a compact metric continuum is locally connected, then it is arcwise connected. However, a semi-locally-connected compact metric continuum, even when lying in the plane, may fail to be arcwise connected [8]. A continuum M is said to be *aposyndetic* at a point p of M with respect to a set N in $M-\{p\}$ if there exist an open set U and a continuum H in M such that $p \in U \subset H \subset M-N$. A continuum M is said to be *aposyndetic* at a point p if for each point q in $M-\{p\}$, M is aposyndetic at p with respect to q. If M is aposyndetic at each of its points, then M is said to be *aposyndetic.* A compact continuum is aposyndetic if and only if it is semi-locally-connected [2]. A plane continuum M is connected im kleinen at a point x of M if and only if for each pair of points y and z in $M-\{x\}$, M is aposyndetic at x with respect to $\{y, z\}$ [4].

In this paper arcwise connectedness is established for certain nonconnected im kleinen aposyndetic compact plane continua. If an aposyndetic compact plane continuum M contains a finite set of points F such that for each point x in $M-F$, there exist two points y and z in F such that M is not aposyndetic at x with respect to $\{y, z\}$, then M is arcwise connected. It is also proved that each point of M is in a simple closed curve which is contained in M and if the set F consists of two points, then M is cyclicly connected (that is, for any points a and b in M, there is a simple closed curve in M which contains $\{a, b\}$). Arcwise connectedness is also established for certain nonaposyndetic plane continua. If a compact plane continuum M is semi-locally-connected at all except a finite number of its points and is such that for each point x in M, M is either not aposyndetic or not semi-locally-connected at x, then M is arcwise connected. For a point x of a continuum M, F. B. Jones defines K_x to be the closed (not necessarily connected) set consisting of x and all points y in $M-\{x\}$ such that M is not aposyndetic at x with respect to y [3, Theorem 2]. Here it is proved that if M is a compact plane continuum such that for each point x of M, the set K_x is finite, and M is either not semi-locally-connected or not aposyndetic at x, then each point of M is in a simple closed curve which is contained in M. Let M be a compact plane continuum which contains a point y such that for each point x in $M-\{y\}$, M is semi-locally-connected at x and

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M is not aposyndetic at x with respect to y. The last result of this paper indicates that if p and q are distinct points of M and no point of M cuts M weakly between p and q, then there is a simple closed curve in M which contains $\{p, q\}$. Examples are given which rule out certain extensions of these results. For definitions of unfamiliar terms and phrases see [5].

2. Aposyndetic continua.

DEFINITION. Let y and z be two distinct points of a compact metric continuum M. Define L_{yz} to be the point set consisting of y, z, and all points x in $M-\{y, z\}$ such that M is not aposyndetic at x with respect to $\{y, z\}$. If M is aposyndetic and there exist distinct points x, y, and z in M such that M is not aposyndetic at x with respect to $\{y, z\}$, then L_{yz} is a subcontinuum of M [6, Theorem 3].

THEOREM 1. *Suppose that M is an aposyndetic compact metric continuum and x, y, and z are distinct points of M such that M is not aposyndetic at x with respect to $\{y, z\}$. Then there exist two disjoint open sets U and V in $M-\{x\}$ containing y and z respectively such that if U' is an open set in U which contains y and V' is an open set in V which contains z, then the x-component of $L_{yz}-(U' \cup V')$ meets both* Bd U' *(the boundary of U') and* Bd V'.

Proof. Since M is aposyndetic at x, there exist continua H and K in $M-\{y\}$ and $M-\{z\}$ respectively such that x is contained in both Int H (the interior of H) and Int K. Define U and V to be disjoint open sets such that $y \in U \subset M-H$ and $z \in V \subset M-K$. Let U' be an open subset of U containing y and let V' be an open subset of V containing z. Let $U_1, U_2, U_3, \ldots$ and $V_1, V_2, V_3, \ldots$ be two monotone descending sequences of circular open sets which are centered on and converge to y and z respectively such that $U_1 \subset U'$ and $V_1 \subset V'$. For each positive integer n, there exists a point x_n in Int $H \cap$ Int K such that (1) the distance from x to x_n is less than $1/n$, (2) the set X_n which is the x_n-component of $M-(U_n \cup V_n)$ meets both Bd U_n and Bd V_n, and does not contain the point x, and (3) for each positive integer m less than n, $X_m \cap X_n = \varnothing$. The limit superior X of $X_1, X_2, X_3, \ldots$ is a continuum which contains $\{x, y, z\}$.

Let w be a point of $X-\{y, z\}$. Suppose that M is aposyndetic at w with respect to $\{y, z\}$. There exist an integer m and a continuum F such that (1) w is in Int F, (2) F is in $M-(U_m \cap V_m)$, and (3) $X_m \cap F \neq \varnothing$. It follows that for each integer n greater than m, the set X_n does not meet F. But this contradicts the fact that w is in the limit superior of $X_1, X_2, X_3, \ldots$. Hence M is not aposyndetic at any point of $X-\{y, z\}$ with respect to $\{y, z\}$.

For each positive integer n, define X_n' to be the x_n-component of $M-(U' \cup V')$. Note that for each n, since x_n is in Int $H \cap$ Int K, X_n' meets both Bd U' and Bd V'. Also note that for each n, X_n' is contained in X_n. Define X' to be the limit superior of $X_1', X_2', X_3', \ldots$. It follows that X' is a subcontinuum of X which meets both Bd U' and Bd V'. Hence the x-component of $L_{yz}-(U' \cup V')$ meets both Bd U' and Bd V'.

DEFINITION. Let M be an aposyndetic compact metric continuum and let x, y, and z be distinct points of M such that M is not aposyndetic at x with respect to $\{y, z\}$. There exist disjoint open sets U and V in $M-\{x\}$ containing y and z respectively such that if U' is an open subset of U which contains y and V' is an open subset of V which contains z, then the x-component of $L_{yz}-(U' \cup V')$ meets both Bd U' and Bd V' (Theorem 1). Let $U_1, U_2, U_3, \ldots$ and $V_1, V_2, V_3, \ldots$ be two monotone descending sequences of circular open sets in M which are centered on and converge to y and z respectively such that $U_1 \subset U$ and $V_1 \subset V$. For each positive integer n, define Y_n to be the x-component of $L_{yz}-(U_n \cup V_n)$. Define L^x_{yz} to be the limit superior of $Y_1, Y_2, Y_3, \ldots$. Since for each n, the continuum Y_n meets both Bd U_n and Bd V_n, L^x_{yz} is a continuum which contains $\{x, y, z\}$.

Throughout this paper S is the set of points of a simple closed surface (that is, a 2-sphere).

THEOREM 2. *If an aposyndetic continuum M in S contains distinct points, x, y, and z such that M is not aposyndetic at x with respect to $\{y, z\}$, then the subcontinuum L^x_{yz} (as previously defined) of M is locally connected.*

Proof. Let $U_1, U_2, U_3, \ldots$ and $V_1, V_2, V_3, \ldots$ be two monotone descending sequences of circular regions in $S-\{x\}$ which are centered on and converge to y and z respectively such that $U_1 \cap V_1 = \varnothing$. As in the preceding definition, for each positive integer n, let Y_n be the x-component of $L_{yz}-(U_n \cup V_n)$. L^x_{yz} is the limit superior of $Y_1, Y_2, Y_3, \ldots$. Suppose that L^x_{yz} is not locally connected. It follows that L^x_{yz} is not connected im kleinen at some point v in $L^x_{yz}-\{x, y, z\}$. There exist two circular regions T and W in S centered on v such that

(1) $T \supset \text{Cl } W$ (Cl W is the closure of W),

(2) $\{x, y, z\} \cap \text{Cl } T = \varnothing$, and

(3) there exists a sequence of mutually exclusive continua $H_1, H_2, H_3, \ldots$ in $L^x_{yz} \cap (\text{Cl } T - W)$ such that each continuum meets both Bd T and Bd W and the limit inferior of the sequence is a continuum [5, Theorem 66, p. 124]. Let q be a point of the limit inferior of $H_1, H_2, H_3, \ldots$ which is not in Bd $T \cup$ Bd W. Let $q_1, q_2, q_3, \ldots$ be a sequence of points converging to q such that for each positive integer n, $q_n \in H_n \cap (T - \text{Cl } W)$. Let $Q_1, Q_2, Q_3, \ldots$ be a sequence of mutually disjoint circular open sets in M which converges to q such that for each positive integer n, the open set Q_n is centered on q_n and Cl Q_n is contained in $T - \text{Cl } W$. For each positive integer n, there exists an integer i such that $Y_i \cap Q_n \neq \varnothing$. It follows that there exists a sequence of mutually exclusive continua $I_1, I_2, I_3, \ldots$ in $L^x_{yz} \cap (\text{Cl } T - W)$ such that for each positive integer n, there exists an integer i such that $I_n \subset Y_i$ and the limit inferior I of $I_1, I_2, I_3, \ldots$ is a continuum which contains q and meets Bd $(\text{Cl } T - W)$. There exist a point p and two circular regions R and E $(R \supset \text{Cl } E)$ centered on q in $T - W$ such that (1) the point p is in $(R - \text{Cl } E) \cap I$ and (2) there exists a sequence of mutually disjoint continua $F_1, F_2, F_3, \ldots$ such that for each positive integer n, there exists an integer i such

that F_n is in $I_i \cap (\mathrm{Cl}\, R-E)$, F_n meets both Bd R and Bd E, and the limit inferior of $F_1, F_2, F_3, \ldots$ is a continuum in L^x_{yz} which contains p.

Assume without loss of generality that the sequence $F_1, F_2, F_3, \ldots$ is such that for each positive integer n, there exist two arc-segments R_n and E_n such that (1) $R_n \subset \mathrm{Bd}\, R$, (2) $E_n \subset \mathrm{Bd}\, E$, and (3) each arc-segment meets $F_1, F_2, F_3, \ldots$ only in F_{2n} and has one endpoint in F_{2n-1} and the other endpoint in F_{2n+1}. Let $p_1, p_2, p_3, \ldots$ be a sequence of points converging to p such that for each positive integer n, the point p_n is in $F_{2n} \cap (R-\mathrm{Cl}\, E)$. Let $P_1, P_2, P_3, \ldots$ be a sequence of circular regions in S such that for each positive integer n, the region P_n is centered on p_n and $\mathrm{Cl}\, P_n$ does not meet $F_{2n-1} \cup F_{2n+1} \cup R_n \cup E_n$. The regions of the sequence $P_1, P_2, P_3, \ldots$ are mutually exclusive and converge to p.

There exist subsequences $U_{n_1}, U_{n_2}, U_{n_3}, \ldots$ and $V_{n_1}, V_{n_2}, V_{n_3}, \ldots$ of $U_1, U_2, U_3, \ldots$ and $V_1, V_2, V_3, \ldots$ respectively such that $\mathrm{Cl}\, U_{n_1} \cap \mathrm{Cl}\, R = \varnothing$, $\mathrm{Cl}\, V_{n_1} \cap \mathrm{Cl}\, R = \varnothing$, and for each positive integer k, the set $F_{2k-1} \cup F_{2k} \cup F_{2k+1}$ is in the x-component of $M-(U_{n_k} \cup V_{n_k})$. The component of $M-(U_{n_1} \cup V_{n_1})$ which contains p_1 is not open relative to M at p_1. Hence the boundary of P_1 contains an arc-segment S_1 whose endpoints a_1 and b_1 lie in different components of $M-(U_{n_1} \cup V_{n_1})$ such that $M \cap S_1 = \varnothing$. There exists a simple closed curve C_1 which separates a_1 from b_1 in S and contains no point of $M-(U_{n_1} \cup V_{n_1})$ such that $C_1 \cap S_1$ is connected. In C_1 there exists an arc-segment T_1 which crosses S_1, contains no point of $M \cup \mathrm{Cl}\, U_{n_1} \cup \mathrm{Cl}\, V_{n_1}$, and has its endpoints in $\mathrm{Bd}\, U_{n_1} \cup \mathrm{Bd}\, V_{n_1}$. The component of $M-(U_{n_2} \cup V_{n_2})$ which contains p_2 is not open relative to M at p_2. Hence the boundary of P_2 contains an arc-segment S_2 whose endpoints a_2 and b_2 lie in different components of $M-(U_{n_2} \cup V_{n_2})$ such that $M \cap S_2 = \varnothing$. There is a simple closed curve C_2 in $U_{n_2} \cup V_{n_2} \cup (S-M)$ which separates a_2 from b_2 in S such that $C_2 \cap S_2$ is connected. In C_2 there is an arc-segment T_2 which crosses S_2, contains no point of $M \cup U_{n_2} \cup V_{n_2}$, and has its endpoints in $\mathrm{Bd}\, U_{n_2} \cup \mathrm{Bd}\, V_{n_2}$. Continue this process. For each positive integer k, there exist two arc-segments S_k and T_k and a simple closed curve C_k such that (1) S_k has endpoints a_k and b_k in M and is contained in $(S-M) \cap \mathrm{Bd}\, P_k$, (2) C_k separates a_k from b_k and contains no point of $M-(U_{n_k} \cup V_{n_k})$, (3) $C_k \cap S_k$ is connected, and (4) T_k is contained in $C_k-(U_{n_k} \cup V_{n_k})$, meets S_k, and has its endpoints in $\mathrm{Bd}\, U_{n_k} \cup \mathrm{Bd}\, V_{n_k}$. For each positive integer k, since $F_{2k-1} \cup F_{2k} \cup F_{2k+1}$ is in the x-component of $M-(U_{n_k} \cup V_{n_k})$, one of the two arcs R_k or E_k must meet points of T_k which precede, and points of T_k which follow $S_k \cap T_k$, with respect to the order of T_k. It follows that for each positive integer k, there exists an arc A_k contained in R_k or contained in E_k such that $A_k \cup T_k$ contains a simple closed curve J_k which separates a_k from b_k in S [5, Theorem 32, p. 181]. The sequence of arcs $A_1, A_2, A_3, \ldots$ has a limit point u in $M \cap \mathrm{Bd}\, (\mathrm{Cl}\, R-E)$. Since M is aposyndetic, there exist a subcontinuum L of M and a region G containing u such that the point p is in $\mathrm{Int}\, L$ and $G \cap L = \varnothing$. There is a positive integer j such that if k is an integer greater than j, then a_k and b_k are in $\mathrm{Int}\, L$. Hence for some integer n, the points a_n and b_n are in L and the arc A_n is contained in G. The simple closed curve J_n separates a_n from b_n in S and does not meet L. This contradicts the fact that L is connected. It follows that L^x_{yz} is locally connected.

THEOREM 3. *If an aposyndetic compact plane continuum M contains a finite set of points F such that for each point x in $M-F$, there exist points y and z in F such that M is not aposyndetic at x with respect to $\{y, z\}$, then M is arcwise connected.*

Proof. Define C to be the finite collection of continua $\{L_{yz} \mid \{y, z\} \subset F$ and L_{yz} is a continuum$\}$. Since M is connected, C is a coherent collection which covers M. Each element of C is arcwise connected (Theorem 2). Therefore M is arcwise connected.

EXAMPLE 1. An aposyndetic compact plane continuum M which contains a countable closed set of points F such that for each point x in $M-F$, there exist two points y and z in F such that M is not aposyndetic at x with respect to $\{y, z\}$, need not be arcwise connected. To see this consider the compact plane continuum M which consists of a simple closed curve J and four sequences of Cantor suspensions $A_1, A_2, A_3, \ldots, B_1, B_2, B_3, \ldots, C_1, C_2, C_3, \ldots$, and $D_1, D_2, D_3, \ldots$(2). The elements of these sequences are joined together at their endpoints and limit on J as indicated in Figure 1.

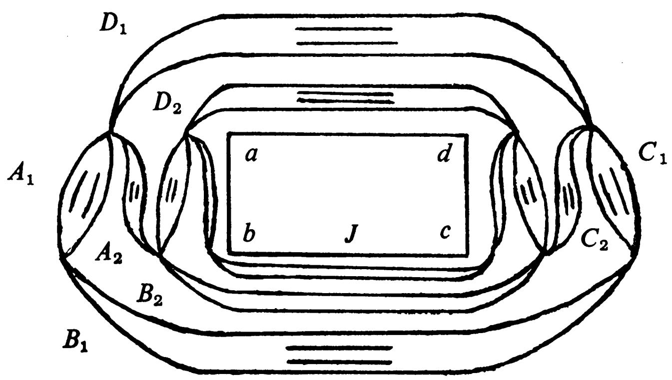

FIGURE 1

Let a, b, c, and d denote the four points of J which are limit points of the set consisting of all endpoints of Cantor suspensions in M. Define F to be the set consisting of a, b, c, d, and all endpoints of Cantor suspensions in M. F is closed and countable, and for each point x in $M-F$, there exist points y and z in F such

(2) A Cantor suspension is a continuum which is the upper semicontinuous decomposition of the topological product of the unit interval [0, 1] and the Cantor discontinuum C in which the sets $0 \times C$ and $1 \times C$ are points. These sets are referred to as endpoints of the Cantor suspension.

that M is not aposyndetic at x with respect to $\{y, z\}$. Since no arc in M meets both J and $M-J$, M is not arcwise connected.

EXAMPLE 2. An aposyndetic compact continuum H in Euclidean 3-space which contains points y and z such that for each point x in $H-\{y, z\}$, H is not aposyndetic at x with respect to $\{y, z\}$, may fail to be arcwise connected. Jones describes a compact plane continuum M consisting of all circles centered on $(\frac{1}{2}, 0)$ and passing through a point of the Cantor discontinuum on the interval from 0 to 1 on the x-axis, together with alternating vertical and horizontal arcs which are each considered to be a point [4, Example 6]. M is aposyndetic and not arcwise connected. Let v and w be distinct points of the circle of largest diameter in M. Let C denote the Cantor discontinuum. Consider the collection of subsets of $M\times C$ which consist of the sets $v\times C$ and $w\times C$, and points of $M\times C-\{v, w\}\times C$. This collection is upper semicontinuous. With respect to its elements as points, this collection is a continuum H which is aposyndetic and can be embedded in Euclidean 3-space. For each point x in $H-\{y, z\}\times C$, H is not aposyndetic at x with respect to $\{y, z\}\times C$. Obviously H is not arcwise connected.

THEOREM 4. *Suppose that M is an aposyndetic compact metric continuum and x, y, and z are distinct points of M such that M is not aposyndetic at x with respect to $\{y, z\}$. Then no point of $L^x_{yz}-\{x, y, z\}$ cuts L^x_{yz} weakly between x and $\{y, z\}$ (that is, for each point p in $L^x_{yz}-\{x, y, z\}$, there is a continuum in $L^x_{yz}-\{p\}$ which meets both x and $\{y, z\}$).*

Proof. Assume that there exists a point q in $L^x_{yz}-\{x, y, z\}$ which cuts L^x_{yz} weakly between x and $\{y, z\}$. Since M is aposyndetic, there exists a continuum H in $M-\{q\}$ which contains x in its interior. Note that H meets $\{y, z\}$. Now $H\cap L_{yz}$ contains no subcontinuum which meets both x and $\{y, z\}$. To see this assume without loss of generality that $H\cap L_{yz}$ contains a subcontinuum L which contains x and y. The subcontinuum of L irreducible from x to y is in L^x_{yz} which contradicts the assumption that q cuts L^x_{yz} weakly between x and $\{y, z\}$. It follows that $H\cap L_{yz}$ is the union of disjoint closed sets A and B containing x and $\{y, z\}\cap H$ respectively. There exist disjoint open sets U and V which contain A and B respectively. $H-(U\cup V)$ is a closed subset of $M-L_{yz}$. Let Q be a finite collection of continua in $M-\{y, z\}$ which cover $H-(U\cup V)$. The set $(U\cap H)\cup \text{St } Q$ (St Q consists of points which are contained in an element of Q) has a finite number of components each of which is closed relative to M. It follows that x is in the interior of one of these components. Since $(U\cap H)\cup \text{St } Q$ is a subset of $M-\{y, z\}$, this is a contradiction. Therefore q does not cut L^x_{yz} weakly between x and $\{y, z\}$.

THEOREM 5. *If an aposyndetic compact plane continuum M contains points w, x, y, and z such that M is not aposyndetic at either w or x with respect to $\{y, z\}$ and w is not in L^x_{yz}, then the continuum $L^w_{yz}\cup L^x_{yz}$ is cyclicly connected.*

Proof. $L^w_{yz}\cup L^x_{yz}$ is locally connected (Theorem 2). Since no point of $\{y, z\}$ separates either L^w_{yz} or L^x_{yz} and $L^w_{yz}\cap L^x_{yz}=\{y, z\}$, no point of $\{y, z\}$ separates

$L^w_{yz} \cup L^x_{yz}$. Suppose there is a point q in $(L^w_{yz} \cup L^x_{yz})-\{y, z\}$ which separates $L^w_{yz} \cup L^x_{yz}$. Assume without loss of generality that q is in L^w_{yz}. Then L^x_{yz} is in a component of $(L^w_{yz} \cup L^x_{yz})-\{q\}$. It follows that q cuts L^w_{yz} between a point of $L^w_{yz}-\{y, z\}$ and $\{y, z\}$. This is impossible (Theorem 4). Hence $L^w_{yz} \cup L^x_{yz}$ does not contain a separating point and is cyclicly connected [7].

THEOREM 6. *If an aposyndetic compact plane continuum M contains points y and z such that for each point x in $M-\{y, z\}$, M is not aposyndetic at x with respect to $\{y, z\}$, then M is cyclicly connected.*

Proof. For any two points a and b in M, there exist points x and w in $M-\{y, z\}$ such that w is not contained in L^x_{yz} and the cyclicly connected continuum $L^w_{yz} \cup L^x_{yz}$ contains $\{a, b\}$.

THEOREM 7. *If an aposyndetic compact plane continuum M contains a finite set of points F such that for each point x in $M-F$, there exist points y and z in F such that M is not aposyndetic at x with respect to $\{y, z\}$, then each point of M is in a simple closed curve which is contained in M.*

Proof. For each point x in M, define H_x to be the finite collection

$$\{L^x_{yz} \mid M \text{ is not aposyndetic at } x \text{ with respect to } \{y, z\} \text{ and } \{y, z\} \subset F-\{x\}\}.$$

For each point x in M, if H_x is nonvoid, then St H_x is a locally connected continuum (Theorem 2) which does not contain an open subset of M. If M is connected im kleinen at x, then H_x is void. Let p be a point of M. There exists an infinite sequence $x_1, x_2, x_3, \ldots$ of points of M such that $x_1=p$ and for each positive integer n, x_{n+1} is in $M-(F \cup \bigcup_{i=1}^n \text{St } H_{x_i})$ and the distance between x_{n+1} and p is less than $1/n$. Suppose there exist a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ of $x_1, x_2, x_3, \ldots$ and points y and z in $F-\{p\}$ such that for each positive integer i, M is not aposyndetic at x_{n_i} with respect to $\{y, z\}$. It follows that M is not aposyndetic at p with respect to $\{y, z\}$. Let x $(x \neq p)$ be a point of $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$. $L^x_{yz} \cup L^p_{yz}$ is cyclicly connected (Theorem 5). Hence there exists a simple closed curve in M which contains p.

If no such subsequence of $x_1, x_2, x_3, \ldots$ exists, then there exist distinct points w and x $(p \notin \{w, x\})$ in $x_1, x_2, x_3, \ldots$ and a point z in F such that M is not aposyndetic at either w or x with respect to $\{p, z\}$. It follows that the locally connected continuum $L^w_{pz} \cup L^x_{pz}$ is cyclicly connected (Theorem 5).

EXAMPLE 3. An aposyndetic compact plane continuum M which contains a countable closed set of points F such that for each point x in $M-F$, there exist points y and z in F such that M is not aposyndetic at x with respect to $\{y, z\}$, may contain a point which is not in a simple closed curve in M. To see this consider a plane continuum M which consists of an arc A with endpoints a and b, and a sequence of mutually exclusive Cantor suspensions $A_1, A_2, A_3, \ldots$ such that (1) for each positive integer n, a_n and b_n are the endpoints of A_n and $A \cap A_n=\{a_n, b_n\}$,

(2) the sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ converge to a and b respectively, and (3) A is the limit set of $A_1, A_2, A_3, \ldots$. Let F be the union of the closures of $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$. For each point x in $M-F$ either there exists a positive integer n such that x is contained in A_n in which case M is not aposyndetic at x with respect to $\{a_n, b_n\}$, or x is contained in A and M is not aposyndetic at x with respect to $\{a, b\}$. No simple closed curve in M meets $\{a, b\}$.

3. **Nonaposyndetic continua.**

DEFINITION. For a point y of a continuum M, Jones defines L_y to be the subcontinuum of M consisting of y and all points x in $M-\{y\}$ such that M is not aposyndetic at x with respect to y [3, Theorem 3]. Let x and y be distinct points of a compact metric continuum M such that M is not aposyndetic at x with respect to y. Let $V_1, V_2, V_3, \ldots$ be a monotone descending sequence of circular open subsets of M ($x \notin V_1$) which are centered on and converge to y. For each positive integer n, let Y_n be the x-component of $L_y - V_n$. Define L_y^x to be the limit superior of $Y_1, Y_2, Y_3, \ldots$. The set L_y^x is a subcontinuum of M which contains x and y. M is not aposyndetic at any point of L_y^x with respect to y.

Recall that for a point z in M, the set K_z consists of z and all points y of $M-\{z\}$ such that M is not aposyndetic at z with respect to y.

THEOREM 8. *Suppose that M is a continuum in S such that for each point z of M, the set K_z is countable. Then if x and y are distinct points of M such that M is not aposyndetic at x with respect to y, the subcontinuum L_y^x (as previously defined) of M is locally connected.*

Proof. Let $V_1, V_2, V_3, \ldots$ be a monotone descending sequence of circular regions in S ($x \notin V_1$) which are centered on and converge to the point y. As in the definition of L_y^x, for each positive integer n, let Y_n be the x-component of $L_y - V_n$. L_y^x is the limit superior of $Y_1, Y_2, Y_3, \ldots$. Assume that L_y^x is not locally connected. It follows that L_y^x is not connected im kleinen at some point p in $L_y^x - \{x, y\}$. There exist two circular regions V and W in S centered on p such that (1) $V \supset \mathrm{Cl}\, W$, (2) $\{x, y\} \cap \mathrm{Cl}\, V = \varnothing$, and (3) there exists a sequence of mutually exclusive continua $H_1, H_2, H_3, \ldots$ in $L_y^x \cap (\mathrm{Cl}\, V - W)$ such that each continuum meets both Bd V and Bd W and the limit inferior of the sequence is a continuum [5, Theorem 66, p. 124]. Let q be a point of the limit inferior of $H_1, H_2, H_2, \ldots$ which is not in Bd $V \cup$ Bd W. Let $q_1, q_2, q_3, \ldots$ be a sequence of points converging to q such that for each positive integer n, $q_n \in H_n \cap (V - \mathrm{Cl}\, W)$. Let $Q_1, Q_2, Q_3, \ldots$ be a sequence of mutually exclusive circular open sets in M which converges to q and has the property that for each positive integer n, the open set Q_n is centered on q_n and Cl Q_n is contained in $V - \mathrm{Cl}\, W$. For each positive integer n, there exists an integer i such that $Y_i \cap Q_n \neq \varnothing$. It follows that there exists a sequence of mutually exclusive continua $I_1, I_2, I_3, \ldots$ in $L_x^y \cap (\mathrm{Cl}\, V - W)$ such that for each positive integer n, there exists an integer i such that $I_n \subset Y_i$ and the limit inferior I of $I_1, I_2, I_3, \ldots$ is a continuum which contains q and meets Bd $(\mathrm{Cl}\, V - W)$. Since the set K_z is countable for each point z in M, there exist a point z and two circular regions R and E ($\mathrm{Cl}\, E \subset R$) centered on q in $V-W$ such that (1) the point z is in $(R-\mathrm{Cl}\, E) \cap I$, (2) M is aposyndetic at z with respect to each point of $M \cap \mathrm{Bd}\,(\mathrm{Cl}\, R-E)$, and (3) there exists a sequence of mutually exclusive continua $F_1, F_2, F_3, \ldots$ such that for each positive integer n, there exists an integer i such that F_n is in $I_i \cap (\mathrm{Cl}\, R-E)$, F_n meets both $\mathrm{Bd}\, R$ and $\mathrm{Bd}\, E$, and the limit inferior of $F_1, F_2, F_3, \ldots$ is a continuum in M which contains z.

Assume without loss of generality that the sequence $F_1, F_2, F_3, \ldots$ is such that for each positive integer n, there exist arc-segments R_n and E_n such that (1) $R_n \subset \mathrm{Bd}\, R$, (2) $E_n \subset \mathrm{Bd}\, E$, and (3) each arc-segment meets $F_1, F_2, F_3, \ldots$ only in F_{2n} and has one endpoint in F_{2n-1} and the other endpoint in F_{2n+1}. Let $z_1, z_2, z_3, \ldots$ be a sequence of points converging to z such that for each positive integer n, the point z_n is in $F_{2n} \cap (R-\mathrm{Cl}\, E)$. Let $Z_1, Z_2, Z_3, \ldots$ be a sequence of circular regions in S such that for each positive integer n, the region Z_n is centered on z_n and $\mathrm{Cl}\, Z_n$ does not meet $F_{2n-1} \cup F_{2n+1} \cup R_n \cup E_n$. Note that the regions of the sequence $Z_1, Z_2, Z_3, \ldots$ are mutually exclusive and converge to z.

There exists a subsequence $V_{n_1}, V_{n_2}, V_{n_3}, \ldots$ of $V_1, V_2, V_3, \ldots$ such that $\mathrm{Cl}\, V_{n_1} \cap \mathrm{Cl}\, R = \varnothing$ and for each positive integer k, the set $F_{2k-1} \cup F_{2k} \cup F_{2k+1}$ is in the x-component of $L_y - V_{n_k}$. The component of $M - V_{n_1}$ which contains z_1 is not open relative to M at z_1. Hence the boundary of Z_1 contains an arc-segment S_1 whose endpoints a_1 and b_1 lie in different components of $M - V_{n_1}$ such that $M \cap S_1 = \varnothing$. There exists a simple closed curve C_1 which separates a_1 from b_1 in S and contains no point of $M - V_{n_1}$ such that $C_1 \cap S_1$ is an arc. In C_1 there exists an arc-segment T_1 which crosses S_1, contains no point of $M \cup V_{n_1}$, and has its endpoints in $\mathrm{Bd}\, V_{n_1}$. The component of $M - V_{n_2}$ which contains z_2 is not open relative to M at z_2. Hence the boundary of Z_2 contains an arc-segment S_2 whose endpoints a_2 and b_2 lie in different components of $M - V_{n_2}$ such that $M \cap S_2 = \varnothing$. There is a simple closed curve C_2 which separates a_2 from b_2 in S and contains no point of $M - V_{n_2}$ such that $C_2 \cap S_2$ is an arc. In C_2 there is an arc-segment T_2 which crosses S_2, contains no point of $M \cup V_{n_2}$, and has its endpoints in $\mathrm{Bd}\, V_{n_2}$. Continue this process. For each positive integer k, there exist two arc-segments S_k and T_k such that (1) S_k has endpoints a_k and b_k in M and is contained in $(S-M) \cap \mathrm{Bd}\, Z_k$, (2) T_k contains no point of $M \cup V_{n_k}$, and (3) $T_k \cup \mathrm{Bd}\, V_{n_k}$ contains a simple closed curve which separates a_k from b_k in S. For each positive integer k, since $F_{2k-1} \cup F_{2k} \cup F_{2k+1}$ is in the x-component of $L_y - V_{n_k}$, one of the two arcs $\mathrm{Cl}\, R_k$ or $\mathrm{Cl}\, E_k$ must meet points of T_k which precede, and points of T_k which follow $S_k \cap T_k$, with respect to the order of T_k. It follows that for each positive integer k, there exists an arc A_k contained in R_k or contained in E_k such that $A_k \cup T_k$ contains a simple closed curve J_k which separates a_k from b_k in S [5, Theorem 32, p. 181]. The sequence of arcs $A_1, A_2, A_3, \ldots$ has a limit point u in $M \cap \mathrm{Bd}\,(\mathrm{Cl}\, R-E)$. Since M is aposyndetic at z with respect to each point of $M \cap \mathrm{Bd}\,(\mathrm{Cl}\, R-E)$, there exist a subcontinuum H of M and a region G in S

containing u such that the point z is in the interior of H and $G \cap H = \varnothing$. There exists a positive integer j such that if k is an integer greater than j, then a_k and b_k are in the interior of H. Hence for some integer n, a_n and b_n are in H and A_n is contained in G. The simple closed curve J_n separates a_n from b_n in S and does not meet H. This contradicts the fact that H is a continuum. It follows that L_y^x must be locally connected.

THEOREM 9. *If M is a compact plane continuum such that M is semi-locally-connected at all except a finite number of its points, and for each point x in M, M is either not aposyndetic at x, or not semi-locally-connected at x, then M is arcwise connected.*

Proof. If M is semi-locally-connected, then M is aposyndetic [2]. Hence there exists a nonvoid finite set of points $\{y_1, y_2, \ldots, y_n\}$ in M such that (1) for each i $(i=1, 2, \ldots, n)$, M is not semi-locally-connected at y_i and L_{y_i} is arcwise connected (Theorem 8), and (2) $\bigcup_{i=1}^{n} L_{y_i} = M$. Since M is the union of a finite number of arcwise connected continua, it is arcwise connected.

COROLLARY. *If M is a totally nonaposyndetic compact plane continuum which is semi-locally-connected at all except a finite number of its points, then M is arcwise connected.*

EXAMPLE 4. A totally nonaposyndetic compact plane continuum which is semi-locally-connected at all except a countable number of its points need not be arcwise connected. To see this consider the compact plane continuum M which consists of two simple closed curves L and J which meet at one point p, and a sequence of Cantor suspensions $S_1, S_2, S_3, \ldots$ (each suspension having its endpoints identified) such that $S_1 \cap L = \{p, q\}$ (q is the endpoint of S_1), and $S_2, S_3, S_4, \ldots$ are joined together and limit on L and J as indicated in Figure 2. M is totally nonaposyndetic and is semi-locally-connected at all points except p, and the endpoints of the elements of $S_1, S_2, S_3, \ldots$. But there does not exist an arc in M which has one endpoint in S_2 and the other in L. Note that for each point x of M, the set K_x consists of only two points.

The next two examples indicate that these results do not hold in Euclidean 3-space.

EXAMPLE 5. A totally nonaposyndetic compact continuum in Euclidean 3-space which is semi-locally-connected at all but a finite number of its points may fail to be arcwise connected. Let M be the plane continuum described by Jones which is aposyndetic and not arcwise connected [4, Example 6]. Let v be a point on the circle of largest diameter in M. Let C denote the Cantor discontinuum. Consider the collection of subsets of $M \times C$ which consists of the set $v \times C$ and points of $M \times C - v \times C$. This collection is upper semicontinuous. With respect to its elements as points, this collection is a continuum H which is semi-locally-connected

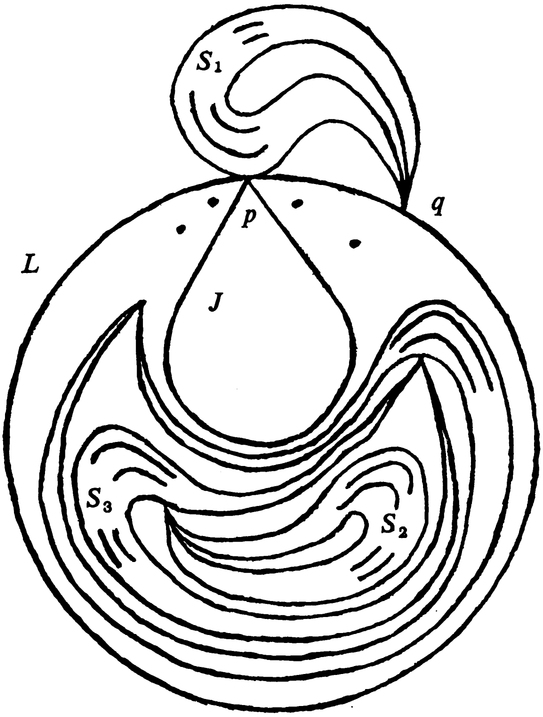

FIGURE 2

at each point of $H-v\times C$ and is not aposyndetic at any point of $H-v\times C$ with respect to the point $v\times C$. H is not arcwise connected. But H is aposyndetic at $v\times C$. However, by appropriately joining two copies of H, it is possible to construct a totally nonaposyndetic compact continuum in Euclidean 3-space which is semi-locally-connected at all but two of its points and is not arcwise connected.

EXAMPLE 6. There exists in Euclidean 3-space a totally nonaposyndetic compact continuum which is semi-locally-connected at all but two of its points and does not contain an arc. To see this let p be a point of a nondegenerate chainable hereditarily indecomposable continuum P (P is the pseudo-arc) and define G to be the collection of subsets of $P\times P$ consisting of the set $p\times P$ and points of $P\times P-p\times P$. G is upper semicontinuous. Since $P\times P$ can be embedded in Euclidean 3-space [1], with respect to its elements as points, G is a continuum M in Euclidean 3-space. M is semi-locally-connected at each point of $M-p\times P$ and is not aposyndetic at any point of $M-p\times P$ with respect to the point $p\times P$. Since P is hereditary indecomposable, M does not contain an arc. M is aposyndetic at the point $p\times P$. However, by appropriately connecting two copies of this continuum, one can construct a totally nonaposyndetic continuum in Euclidean 3-space which is semi-locally-connected at all but two of its points and does not contain an arc.

THEOREM 10. *Suppose that x and y are points of a compact metric continuum M such that M is not aposyndetic at x with respect to y. Then if a point q of $L_y^x-\{x, y\}$ cuts M weakly between x and y, M is not aposyndetic at x with respect to q.*

Proof. Assume that a point q of $L_y^x-\{x, y\}$ cuts M weakly between x and y, and M is aposyndetic at x with respect to q. There exists a continuum H in $M-\{q\}$ which contains x in its interior. $H \cap L_y$ contains no subcontinuum containing both x and y. To see this suppose there is a subcontinuum L of $H \cap L_y$ which contains $\{x, y\}$. Let I be a subcontinuum of L which is irreducible from x to y. I is contained in L_y^x. Therefore q is in I. Since H is in $M-\{q\}$, this is impossible. It follows that $H \cap L_y$ is the union of disjoint closed sets A and B containing x and y respectively. There exist disjoint open sets U and V which contain A and B respectively. $H-(U \cup V)$ is a closed subset of $M-L_y$. Let Q be a finite collection of continua in $M-\{y\}$ which covers $H-(U \cup V)$. The set $(U \cap H) \cup \operatorname{St} Q$ has a finite number of components each of which is closed relative to M. It follows that x is in the interior of one of these components. Since this component is contained in $M-\{y\}$, this is a contradiction. Hence M is not aposyndetic at x with respect to q.

THEOREM 11. *If M is a compact plane continuum such that for each point x in M, the set K_x is finite, and M is either not semi-locally-connected at x or not aposyndetic at x, then each point of M is in a simple closed curve which is contained in M.*

Proof. If p is a point of M and M is not semi-locally-connected at p, then there exists a point x in M such that M is not aposyndetic at x with respect to p. Since the continuum L_p^x is locally connected (Theorem 8) and K_x is finite, there exists a point q in L_p^x such that no point of L_p^x separates p from q in L_p^x (Theorem 10). It follows that L_p^x contains a simple closed curve which contains p and q [7]. If M is semi-locally-connected at p, then there exists a point y in M such that M is not aposyndetic at p with respect to y. It follows from the same argument that in L_y^p there is a simple closed curve which contains p.

Note that the continuum M in Example 4 has all the properties stated in Theorem 11.

COROLLARY. *If M is a totally nonaposyndetic compact plane continuum which is semi-locally-connected at all except a finite number of its points, then each point of M is in a simple closed curve which is contained in M.*

EXAMPLE 7. If M is a totally nonaposyndetic compact plane continuum such that for each point x in M, the set K_x is countable, then each point of M need not be contained in a simple closed curve. To see this consider the plane continuum M consisting of four sequences of Cantor suspensions $R_1, R_2, R_3, \ldots, S_1, S_2, S_3, \ldots, T_1, T_2, T_3, \ldots$, and $Y_1, Y_2, Y_3, \ldots$ (each suspension having its endpoints identified), and a point p such that (1) $S_1, S_2, S_3, \ldots$ and $T_1, T_2, T_3, \ldots$ converge to p, (2) for each positive integer i, R_i and Y_i have the same endpoint, and (3) the sets

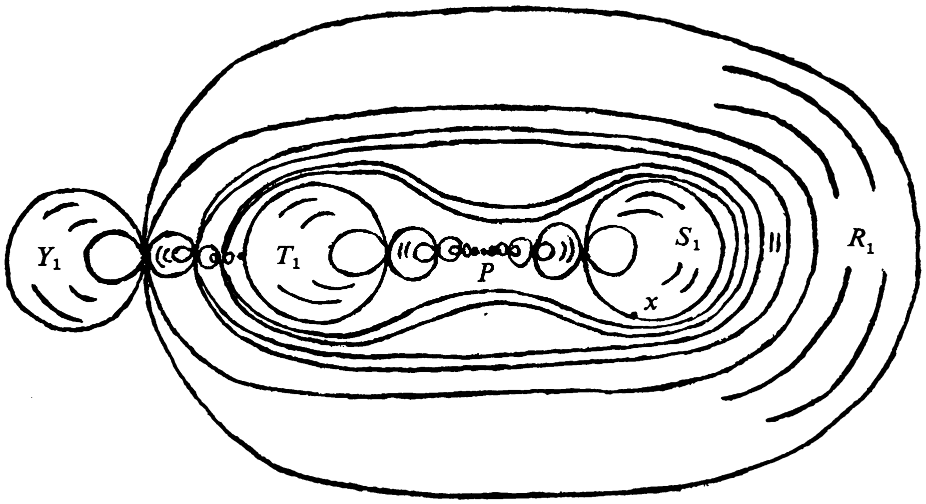

FIGURE 3

$R_1, R_2, R_3, \ldots$ loop around and limit on a subset of $\bigcup_i S_i \cup T_i$ as indicated in Figure 3. M is totally nonaposyndetic. The point p is not in a simple closed curve contained in M. Note that if x is a point of S_1 which is in the limit set of $R_1, R_2, R_3, \ldots$ then the set K_x is countably infinite.

THEOREM 12. *Suppose that M is a compact plane continuum which contains a point y such that for each point x in $M-\{y\}$, M is semi-locally-connected at x and M is not aposyndetic at x with respect to y. Then if p and q are distinct points of M and no point cuts M weakly between p and q, there exists a simple closed curve in M which contains $\{p, q\}$.*

Proof. Assume without loss of generality that $p \neq y$. The point q is in L_y^p for if it were not y would cut M weakly between p and q. L_y^p is locally connected (Theorem 8) and does not contain a separating point (Theorem 10). Hence in L_y^p there is a simple closed curve which contains $\{p, q\}$.

The following example indicates that the words "cuts M weakly" in the statement of Theorem 12 cannot be replaced by "separates M".

EXAMPLE 8. Let I be the unit interval and let C be the Cantor discontinuum in I. For each point r in I, let $V(r)$ be the vertical line $\{(r, y) \mid y \in I\}$ in $I \times I$. For each positive integer n, define T_n to be the collection

$$\{\text{Cl } H \mid H \text{ is an open segment in } (V(1/3n) \cup V(1-1/3n)) - I \times C \text{ which has both endpoints in } I \times C \text{ and diameter } 1/3n\}.$$

Let G be the collection consisting of $\bigcup_{n=1}^{\infty} T_n$, the set $V(0) \cup V(1)$, and points of $I \times C - (V(0) \cup V(1) \cup \bigcup_{n=1}^{\infty} \text{St } T_n)$. G is upper semicontinuous. With respect to its elements as points, G is a continuum M. For each point x in $M-(V(0) \cup V(1))$,

M is semi-locally-connected at x and is not aposyndetic at x with respect to the point $V(0) \cup V(1)$. M does not contain a separating point and is not cyclicly connected. Note that $V(0) \cup V(1)$ is a weak cut point in M.

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